ON CARLITZ'S TYPE q-EULER NUMBERS ASSOCIATED WITH THE FERMIONIC p-ADIC INTEGRAL ON $\mathbb{Z}_p$

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Abstract. In this paper, we consider the following problem in [20]: "Find Witt's formula for Carlitz's type q-Euler numbers." We give Witt's formula for Carlitz's type q-Euler numbers, which is an answer to the above problem. Moreover, we obtain a new p-adic q-l-function $l_{p,q}(s,\chi)$ for Dirchlet's character χ, with the property that

$$l_{p,q}(-n,\chi)=E_{n,\chi_n,q}-\chi_n(p)[p]_q^n E_{n,\chi_n,q^p},\quad n=0,1,\dots$$

using the fermionic p-adic integral on $\mathbb{Z}_p$.

1. Introduction

Throughout this paper, let p be an odd prime number. The symbol $\mathbb{Z}_p,\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the rings of p-adic integers, the field of p-adic numbers and the field of p-adic completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic absolute value in $\mathbb{C}_p$ is normalized in such way that $|p|_p=p^{-1}$. Let $\mathbb{N}$ be the natural numbers and $\mathbb{Z}^+=\mathbb{N}\cup\{0\}$.

As the definition of q-number, we use the following notations:

$$[x]_q=\frac{1-q^x}{1-q}\quad\text{and}\quad[x]_{-q}=\frac{1-(-q)^x}{1+q}.$$

Note that $\lim_{q\to1}[x]_q=x$ for $x\in\mathbb{Z}_p$, where q tends to 1 in the region $0<|q-1|_p<1$.

When one talks of q-analogue, q is variously considered as an indeterminate, a complex number $q\in\mathbb{C}$, or a p-adic number $q\in\mathbb{C}_p$. If $q=1+t\in\mathbb{C}_p$, one normally assumes $|t|_p<1$. We shall further suppose that $\operatorname{ord}_p(t)>1/(p-1)$, so that $q^x=\exp(x\log q)$ for $|x|_p\le1$. If $q\in\mathbb{C}$, then we assume that $|q|<1$.

After Carlitz [3, 4] gave q-extensions of the classical Bernoulli numbers and polynomials, the q-extensions of Bernoulli and Euler numbers and polynomials have been studied by several authors (cf. [1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24]). The Euler numbers and polynomials have been studied by researchers in the field of number theory, mathematical physics and so on (cf. [3, 4, 5, 11, 13, 15, 16, 17, 18, 23]). Recently, various q-extensions of these numbers and polynomials have been studied by many mathematicians (cf. [8, 9, 10, 12, 14, 19, 20, 22]). Also, some authors have studied in the several area of q-theory (cf. [1, 2, 6, 18, 21]).

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It is known that the generating function of Euler numbers $F(t)$ is given by

$$F(t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1.1}$$

From (1.1), we known that the recurrence formula of Euler numbers is given by

$$E_0 = 1, \qquad (E+1)^n + E_n = 0 \quad \text{if } n > 0 \tag{1.2}$$

with the usual convention of replacing E^n by E_n (see [9, 20]).

In [19], the q-extension of Euler numbers $E^*_{n,q}$ are defined as

$$E^*_{0,q} = 1, \qquad (qE^* + 1)^n + E^*_{n,q} = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases} \tag{1.3}$$

with the usual convention of replacing $(E^*)^n$ by $E^*_{n,q}$.

As the same motivation of the construction in [20], Carlitz's type q-Euler numbers $E_{n,q}$ are defined as

$$E_{0,q} = \frac{2}{[2]_q}, \qquad q(qE+1)^n + E_{n,q} = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases} \tag{1.4}$$

with the usual convention of replacing E^n by $E_{n,q}$. It was shown that $\lim_{q\to 1} E_{n,q} = E_n$, where E_n is the nth Euler number. In the complex case, the generating function of Carlitz's type q-Euler numbers $F_q(t)$ is given by

$$F_q(t) = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!} = 2\sum_{n=0}^{\infty} (-q)^n e^{[n]_q t}, \tag{1.5}$$

where q is complex number with $|q| < 1$ (see [20]). The remark point is that the series on the right-hand side of (1.5) is uniformly convergent in the wider sense. In p-adic case, Kim et al. [20] could not determine the generating function of Carlitz's type q-Euler numbers and Witt's formula for Carlitz's type q-Euler numbers.

In this paper, we obtain the generating function of Carlitz's type q-Euler numbers in the p-adic case. Also, we give Witt's formula for Carlitz's type q-Euler numbers, which is a partial answer to the problem in [20]. Moreover, we obtain a new p-adic q-l-function $l_{p,q}(s,\chi)$ for Dirchlet's character χ, with the property that

$$l_{p,q}(-n,\chi) = E_{n,\chi_n,q} - \chi_n(p)[p]_q^n E_{n,\chi_n,q^p}$$

for $n \in \mathbb{Z}^+$ using the fermionic p-adic integral on $\mathbb{Z}_p$.

2. Carlitz's type q-Euler numbers in the p-adic case

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. Then the p-adic q-integral of a function $f \in UD(\mathbb{Z}_p)$ on $\mathbb{Z}_p$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(a)d\mu_q(a) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{a=0}^{p^N-1} f(a)q^a \tag{2.1}$$

(cf. [5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22]). The bosonic p-adic integral on $\mathbb{Z}_p$ is considered as the limit $q \to 1$, i.e.,

$$I_1(f) = \int_{\mathbb{Z}_p} f(a)d\mu_1(a). \tag{2.2}$$

From (2.1), we have the fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f) = \lim_{q \to -1} I_q(f) = \int_{\mathbb{Z}_p} f(a) d\mu_{-1}(a). \tag{2.3}$$

Using formula (2.3), we can readily derive the classical Euler polynomials, $E_n(x)$, namely

$$2\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{2.4}$$

In particular when $x = 0$, $E_n(0) = E_n$ is well known the Euler numbers (cf. [9, 18, 21]).

By definition of $I_{-1}(f)$, we show that

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{2.5}$$

where $f_1(x) = f(x+1)$ (see [9]). By (2.5) and induction, we obtain the following fermionic p-adic integral equation

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2\sum_{i=0}^{n-1} (-1)^{n-i-1} f(i), \tag{2.6}$$

where $n = 1, 2, \ldots$ and $f_n(x) = f(x+n)$. From (2.6), we note that

$$I_{-1}(f_n) + I_{-1}(f) = 2\sum_{i=0}^{n-1} (-1)^i f(i) \quad \text{if } n \text{ is odd;} \tag{2.7}$$

$$I_{-1}(f_n) - I_{-1}(f) = 2\sum_{i=0}^{n-1} (-1)^{i+1} f(i) \quad \text{if } n \text{ is even.} \tag{2.8}$$

For $x \in \mathbb{Z}_p$ and any integer $i \geq 0$, we define

$$\binom{x}{i} = \begin{cases} \frac{x(x-1)\cdots(x-i+1)}{i!} & \text{if } i \geq 1, \\ 1, & \text{if } i = 0. \end{cases} \tag{2.9}$$

It is easy to see that $\binom{x}{i} \in \mathbb{Z}_p$ (see [23, p. 172]). We put $x \in \mathbb{C}_p$ with $\operatorname{ord}_p(x) > 1/(p-1)$ and $|1-q|_p < 1$. We define q^x for $x \in \mathbb{Z}_p$ by

$$q^x = \sum_{i=0}^{\infty} \binom{x}{i} (q-1)^i \quad \text{and} \quad [x]_q = \sum_{i=1}^{\infty} \binom{x}{i} (q-1)^{i-1}. \tag{2.10}$$

If we set $f(x) = q^x$ in (2.7) and (2.8), we have

$$I_{-1}(q^x) = \frac{2}{q^n+1} \sum_{i=0}^{n-1} (-1)^i q^i = \frac{2}{q+1} \quad \text{if } n \text{ is odd;} \tag{2.11}$$

$$I_{-1}(q^x) = \frac{2}{q^n-1} \sum_{i=0}^{n-1} (-1)^{i+1} q^i = \frac{2}{q+1} \quad \text{if } n \text{ is even.} \tag{2.12}$$

Thus for each $l \in \mathbb{N}$ we obtain $I_{-1}(q^{lx}) = \frac{2}{q^l+1}$. Therefore we have

$$
\begin{aligned}
I_{-1}(q^x[x]_q^n) &= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-l)^l I_{-1}(q^{(l+1)x}) \\
&= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-l)^l \frac{2}{q^{l+1}+1}.
\end{aligned} \tag{2.13}
$$

Also, if $f(x) = q^{lx}$ in (2.5), then

$$
I_{-1}(q^{l(x+1)}) + I_{-1}(q^{lx}) = 2f(0) = 2. \tag{2.14}
$$

On the other hand, by (2.14), we obtain that

$$
\begin{aligned}
&I_{-1}(q^{x+1}[x+1]_q^n) + I_{-1}(q^x[x]_q^n) \\
&= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l \left(I_{-1}((q^{l+1})^{x+1}) + I_{-1}((q^{l+1})^x)\right) \\
&= \frac{2}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l \\
&= 0
\end{aligned} \tag{2.15}
$$

is equivalent to

$$
\begin{aligned}
0 &= I_{-1}(q^{x+1}[x+1]_q^n) + I_{-1}(q^x[x]_q^n) \\
&= qI_{-1}(q^x(1+q[x]^n)) + I_{-1}(q^x[x]_q^n) \\
&= qI_{-1}\left(q^x\sum_{l=0}^{n}\binom{n}{l}q^l[x]^l\right) + I_{-1}(q^x[x]_q^n) \\
&= q\sum_{l=0}^{n}\binom{n}{l}q^l I_{-1}(q^x[x]^l) + I_{-1}(q^x[x]_q^n).
\end{aligned} \tag{2.16}
$$

From the definition of fermionic p-adic integral on $\mathbb{Z}_p$ and (2.13), we can derive the following formula

$$
\begin{aligned}
I_{-1}(q^x[x]_q^n) &= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}[x]_q^n q^x d\mu_{-1}(x) \\
&= \lim_{N\to\infty}\sum_{a=0}^{p^N-1}\frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^i q^{ia}(-q)^a \\
&= \frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^i \lim_{N\to\infty}\sum_{a=0}^{p^N-1}(-1)^a(q^{i+1})^a \\
&= \frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^i \frac{2}{1+q^{i+1}}
\end{aligned} \tag{2.17}
$$

is equivalent to

$$\sum_{n=0}^{\infty} I_{-1}(q^x[x]_q^n)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\frac{2}{1+q^{i+1}}\frac{t^n}{n!} \tag{2.18}$$
$$= 2\sum_{n=0}^{\infty}(-q)^n e^{[n]_q t}.$$

From (2.14), (2.15), (2.16), (2.17) and (2.18), it is easy to show that

$$q\sum_{l=0}^{n}\binom{n}{l}q^l E_{l,q} + E_{n,q} = \begin{cases} 2 & \text{if } n=0 \\ 0 & \text{if } n>0, \end{cases} \tag{2.19}$$

where $E_{n,q}$ are Carlitz's type q-Euler numbers defined by (see [20])

$$F_q(t) = 2\sum_{n=0}^{\infty}(-q)^n e^{[n]_q t} = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}. \tag{2.20}$$

Therefore, we obtain the recurrence formula for the Carlitz's type q-Euler numbers as follows:

$$q(qE+1)^n + E_{n,q} = \begin{cases} 2 & \text{if } n=0 \\ 0 & \text{if } n>0 \end{cases} \tag{2.21}$$

with the usual convention of replacing E^n by $E_{n,q}$. Therefore, by (2.18), (2.20) and (2.21), we obtain the following theorem.

Theorem 2.1 (Witt's formula for $E_{n,q}$)**.** *For $n \in \mathbb{Z}^+$,*

$$E_{n,q} = \frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\frac{2}{1+q^{i+1}} = \int_{\mathbb{Z}_p}[x]_q^n q^x d\mu_{-1}(x),$$

which is a partial answer to the problem in [20]*. Carlitz's type q-Euler numbers $E_n = E_{n,q}$ can be determined inductively by*

$$q(qE+1)^n + E_{n,q} = \begin{cases} 2 & \textit{if } n=0 \\ 0 & \textit{if } n>0 \end{cases} \tag{2.22}$$

with the usual convention of replacing E^n by $E_{n,q}$.

Carlitz type q-Euler polynomials $E_{n,q}(x)$ are defined by means of the generating function $F_q(x,t)$ as follows:

$$F_q(x,t) = 2\sum_{k=0}^{\infty}(-1)^k q^k e^{[k+x]_q t} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}. \tag{2.23}$$

In the cases $x=0$, $E_{n,q}(0) = E_{n,q}$ will be called Carlitz type q-Euler numbers (cf. [10, 21]). We also can see that the generating functions $F_q(x,t)$ are determined as solutions of the following q-difference equation:

$$F_q(x,t) = 2e^{[x]_q t} - qe^t F_q(x,qt). \tag{2.24}$$

From (2.23), we get the following:

Lemma 2.2. (1) $F_q(x,t) = 2e^{\frac{t}{1-q}}\sum_{j=0}^{\infty}\left(\frac{1}{q-1}\right)^j q^{xj}\frac{1}{1+q^{j+1}}\frac{t^j}{j!}$.

(2) $E_{n,q}(x) = 2\sum_{k=0}^{\infty}(-1)^k q^k [k+x]_q^n$.

It is clear from (1) and (2) of Lemma 2.2 that

$$E_{n,q}(x) = \frac{2}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{1+q^{k+1}} q^{xk} \tag{2.25}$$

and

$$\begin{aligned} \sum_{k=0}^{m-1} (-1)^k q^k [k+x]_q^n &= \sum_{k=0}^{\infty} (-1)^k q^k [k+x]_q^n \\ &\quad - \sum_{k=0}^{\infty} (-1)^{k+m} q^{k+m} [k+m+x]_q^n \\ &= \frac{1}{2} \left(E_{n,q}(x) + (-1)^{m+1} q^m E_{n,q}(x+m) \right). \end{aligned} \tag{2.26}$$

From (2.25) and (2.26), we may state

Proposition 2.3. *If $m \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, then*

(1) $E_{n,q}(x) = \frac{2}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{1+q^{k+1}} q^{xk}$.
(2) $\sum_{k=0}^{m-1} (-1)^k q^k [k+x]_q^n = \frac{1}{2} \left(E_{n,q}(x) + (-1)^{m+1} q^m E_{n,q}(x+m) \right)$.

Proposition 2.4. *For $n \in \mathbb{Z}^+$, the value of $\int_{\mathbb{Z}_p} [x+y]_q^n q^y d\mu_{-1}(y)$ is $n!$ times the coefficient of t^n in the formal expansion of $2\sum_{k=0}^{\infty}(-1)^k q^k e^{[k+x]_q t}$ in powers of t. That is, $E_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n q^y d\mu_{-1}(y)$.*

Proof. From (2.3), we have the relation

$$\int_{\mathbb{Z}_p} q^{k(x+y)} q^y d\mu_{-1}(y) = q^{xk} \lim_{N\to\infty} \sum_{a=0}^{p^N-1} (-q^{k+1})^a = \frac{2q^{xk}}{1+q^{k+1}}$$

which leads to

$$\begin{aligned} \int_{\mathbb{Z}_p} [x+y]_q^n q^y d\mu_{-1}(y) &= 2 \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(1-q)^n} (-1)^k \int_{\mathbb{Z}_p} q^{k(x+y)} q^y d\mu_{-1}(y) \\ &= \frac{2}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{1+q^{k+1}} q^{xk}. \end{aligned}$$

The result now follows by using (1) of Proposition 2.3. □

Corollary 2.5. *If $n \in \mathbb{Z}^+$, then*

$$E_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} E_{k,q}.$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$ and p be a fixed odd prime number. We set

$$\begin{gathered} X = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \quad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \\ a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{gathered} \tag{2.27}$$

where $a \in \mathbb{Z}$ with $0 \le a < dp^N$ (cf. [9, 11]). Note that the natural map $\mathbb{Z}/dp^N\mathbb{Z} \to \mathbb{Z}/p^N\mathbb{Z}$ induces

$$\pi : X \to \mathbb{Z}_p. \tag{2.28}$$

Hereafter, if f is a function on $\mathbb{Z}_p$, we denote by the same f the function $f \circ \pi$ on X. Namely we consider f as a function on X.

Let χ be the Dirichlet's character with an odd conductor $d = d_\chi \in \mathbb{N}$. Then the generalized Carlitz type q-Euler polynomials attached to χ defined by

$$E_{n,\chi,q}(x) = \int_X \chi(a)[x+y]_q^n q^y d\mu_{-1}(y), \tag{2.29}$$

where $n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}_p$. Then we have the generating function of generalized Carlitz type q-Euler polynomials attached to χ :

$$F_{q,\chi}(x,t) = 2\sum_{m=0}^{\infty} \chi(m)(-1)^m q^m e^{[m+x]_q t} = \sum_{n=0}^{\infty} E_{n,\chi,q}(x)\frac{t^n}{n!}. \tag{2.30}$$

Now fixed any $t \in \mathbb{C}_p$ with $\mathrm{ord}_p(t) > 1/(p-1)$ and $|1-q|_p < 1$. From (2.30), we have

$$\begin{aligned} F_{q,\chi}(x,t) &= 2\sum_{m=0}^{\infty} \chi(m)(-q)^m \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i}(-1)^i q^{i(m+x)} \frac{t^n}{n!} \\ &= 2\sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i}(-1)^i q^{ix} \\ &\quad \times \sum_{j=0}^{d-1}\sum_{l=0}^{\infty} \chi(j+dl)(-q)^{j+dl} q^{i(j+dl)} \frac{t^n}{n!} \\ &= 2\sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \sum_{j=0}^{d-1} \chi(j)(-q)^j \sum_{i=0}^{n} \binom{n}{i}(-1)^i \frac{q^{i(x+j)}}{1+q^{d(i+1)}} \frac{t^n}{n!}, \end{aligned} \tag{2.31}$$

where $x \in \mathbb{Z}_p$ and $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. By (2.30) and (2.31), we can derive the following formula

$$\begin{aligned} E_{n,\chi,q}(x) &= \frac{1}{(1-q)^n} \sum_{j=0}^{d-1} \chi(j)(-q)^j \sum_{i=0}^{n} \binom{n}{i}(-1)^i q^{i(x+j)} \frac{2}{1+q^{d(i+1)}} \\ &= \frac{1}{(1-q)^n} \sum_{j=0}^{d-1} \chi(j)(-q)^j \sum_{i=0}^{n} \binom{n}{i}(-1)^i q^{i(x+j)} \\ &\quad \times \lim_{N\to\infty} \sum_{l=0}^{p^N-1} (-1)^l (q^{d(i+1)})^l \\ &= \lim_{N\to\infty} \sum_{j=0}^{d-1}\sum_{l=0}^{p^N-1} \chi(j+dl) \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i}(-1)^i q^{i(j+dl+x)} \\ &\quad \times (-1)^{j+dl} q^{j+dl} \\ &= \lim_{N\to\infty} \sum_{a=0}^{dp^N-1} \chi(a) \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i}(-1)^i q^{i(a+x)}(-q)^a \\ &= \int_X \chi(y)[x+y]_q^n q^y d\mu_{-1}(y), \end{aligned} \tag{2.32}$$

where $x \in \mathbb{Z}_p$ and $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Therefore, we obtain the following

Theorem 2.6.

$$E_{n,\chi,q}(x) = \frac{1}{(1-q)^n} \sum_{j=0}^{d-1} \chi(j)(-q)^j \sum_{i=0}^{n} \binom{n}{i} (-1)^i q^{i(x+j)} \frac{2}{1+q^{d(i+1)}},$$

where $n \in \mathbb{Z}^+$ *and* $x \in \mathbb{Z}_p$.

Let ω denote the Teichmüller character mod p. For $x \in X^*$, we set

$$\langle x \rangle = [x]_q \omega^{-1}(x) = \frac{[x]_q}{\omega(x)}. \tag{2.33}$$

Note that since $|\langle x \rangle - 1|_p < p^{-1/(p-1)}$, $\langle x \rangle^s$ is defined by $\exp(s \log_p \langle x \rangle)$ for $|s|_p \leq 1$ (cf. [12, 14, 24]). We note that $\langle x \rangle^s$ is analytic for $s \in \mathbb{Z}_p$.

We define an interpolation function for Carlitz type q-Euler numbers. For $s \in \mathbb{Z}_p$,

$$l_{p,q}(s,\chi) = \int_{X^*} \langle x \rangle^{-s} \chi(x) q^x d\mu_{-1}(x). \tag{2.34}$$

Then $l_{p,q}(s,\chi)$ is analytic for $s \in \mathbb{Z}_p$.

The values of this function at non-positive integers are given by

Theorem 2.7. *For integers* $n \geq 0$,

$$l_{p,q}(-n,\chi) = E_{n,\chi_n,q} - \chi_n(p)[p]_q^n E_{n,\chi_n,q^p},$$

where $\chi_n = \chi\omega^{-n}$. *In particular, if* $\chi = \omega^n$, *then* $l_{p,q}(-n,\omega^n) = E_{n,q} - [p]_q E_{n,q^p}$.

Proof.

$$\begin{aligned} l_{p,q}(-n,\chi) &= \int_{X^*} \langle x \rangle^n \chi(x) q^x d\mu_{-1}(x) \\ &= \int_X [x]_q^n \chi_n(x) q^x d\mu_{-1}(x) - \int_X [px]_q^n \chi_n(px) q^{px} d\mu_{-1}(px) \\ &= \int_X [x]_q^n \chi_n(x) q^x d\mu_{-1}(x) - [p]_q^n \chi_n(p) \int_X [x]_{q^p}^n \chi_n(x) q^{px} d\mu_{-1}(x). \end{aligned}$$

Therefore by (2.29), the theorem is proved. □

Let χ be the Dirichlet's character with an odd conductor $d = d_\chi \in \mathbb{N}$. Let F be a positive integer multiple of p and d. Then by (2.23) and (2.30), we have

$$\begin{aligned} F_{q,\chi}(x,t) &= 2 \sum_{m=0}^{\infty} \chi(m)(-1)^m q^m e^{[m+x]_q t} \\ &= 2 \sum_{a=0}^{F-1} \chi(a)(-q)^a \sum_{k=0}^{\infty} (-q)^{Fk} e^{[F]_q \left[k+\frac{x+a}{F}\right]_{q^F} t} \\ &= \sum_{n=0}^{\infty} \left([F]_q^n \sum_{a=0}^{F-1} \chi(a)(-q)^a E_{n,q^F}\left(\frac{x+a}{F}\right)\right) \frac{t^n}{n!}. \end{aligned} \tag{2.35}$$

Therefore we obtain the following

$$E_{n,\chi,q}(x) = [F]_q^n \sum_{a=0}^{F-1} \chi(a)(-q)^a E_{n,q^F}\left(\frac{x+a}{F}\right). \tag{2.36}$$

If $\chi_n(p) \neq 0$, then $(p, d_{\chi_n}) = 1$, so that F/p is a multiple of d_{χ_n}. From (2.36), we derive

$$\begin{aligned}\chi_n(p)[p]_q^n E_{n,\chi_n,q^p} &= \chi_n(p)[p]_q^n [F/p]_{q^p}^n \sum_{a=0}^{F/p-1} \chi_n(a)(-q^p)^a E_{n,(q^p)^{F/p}}\left(\frac{a}{F/p}\right) \\ &= [F]_q^n \sum_{\substack{a=0\\ p\mid a}}^{F} \chi_n(a)(-q)^a E_{n,q^F}\left(\frac{a}{F}\right).\end{aligned} \tag{2.37}$$

Thus we have

$$E_{n,\chi_n,q} - \chi_n(p)[p]_q^n E_{n,\chi_n,q^p} = [F]_q^n \sum_{\substack{a=0\\ p\nmid a}}^{F-1} \chi_n(a)(-q)^a E_{n,q^F}\left(\frac{a}{F}\right). \tag{2.38}$$

By Corollary 2.5, we easily see that

$$\begin{aligned}E_{n,q^F}\left(\frac{a}{F}\right) &= \sum_{k=0}^{n} \binom{n}{k} \left[\frac{a}{F}\right]_{q^F}^{n-k} q^{ka} E_{k,q^F} \\ &= [F]_q^{-n}[a]_q^n \sum_{k=0}^{n} \binom{n}{k} \left[\frac{F}{a}\right]_{q^a}^{k} q^{ka} E_{k,q^F}.\end{aligned} \tag{2.39}$$

From (2.38) and (2.39), we have

$$\begin{aligned}&E_{n,\chi_n,q} - \chi_n(p)[p]_q^n E_{n,\chi_n,q^p} \\ &\quad= [F]_q^n \sum_{\substack{a=0\\ p\nmid a}}^{F-1} \chi_n(a)(-q)^a E_{n,q^F}\left(\frac{a}{F}\right) \\ &\quad= \sum_{\substack{a=0\\ p\nmid a}}^{F-1} \chi(a)\langle a\rangle^n(-q)^a \sum_{k=0}^{\infty} \binom{n}{k} \left[\frac{F}{a}\right]_{q^a}^{k} q^{ka} E_{k,q^F},\end{aligned} \tag{2.40}$$

since $\chi_n(a) = \chi(a)\omega^{-n}(a)$. From Theorem 2.7 and (2.40),

$$l_{p,q}(-n,\chi) = \sum_{\substack{a=0\\ p\nmid a}}^{F-1} \chi(a)\langle a\rangle^n(-q)^a \sum_{k=0}^{\infty} \binom{n}{k} \left[\frac{F}{a}\right]_{q^a}^{k} q^{ka} E_{k,q^F} \tag{2.41}$$

for $n \in \mathbb{Z}^+$. Therefore we have the following theorem.

Theorem 2.8. *Let F be a positive integer multiple of p and $d = d_\chi$, and let*

$$l_{p,q}(s,\chi) = \int_{X^*} \langle x\rangle^{-s}\chi(x)q^x \, d\mu_{-1}(x), \quad s \in \mathbb{Z}_p.$$

Then $l_{p,q}(s,\chi)$ is analytic for $s \in \mathbb{Z}_p$ and

$$l_{p,q}(s,\chi) = \sum_{\substack{a=0\\ p\nmid a}}^{F-1} \chi(a)\langle a\rangle^{-s}(-q)^a \sum_{k=0}^{\infty} \binom{-s}{k} \left[\frac{F}{a}\right]_{q^a}^{k} q^{ka} E_{k,q^F}.$$

Furthermore, for $n \in \mathbb{Z}^+$

$$l_{p,q}(-n,\chi) = E_{n,\chi_n,q} - \chi_n(p)[p]_q^n E_{n,\chi_n,q^p}.$$

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